

ON THE PARTIAL SUMS OF VILENKIN-FOURIER SERIES

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ABSTRACT. The main aim of this paper is to investigate weighted maximal operators of partial sums of Vilenkin-Fourier series. We also use our results to prove approximation and strong convergence theorems on the martingale Hardy spaces H_p , when $0 < p \leq 1$.

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1. INTRODUCTION

It is well-known that Vilenkin system forms not basis in the space $L_1(G_m)$. Moreover, there is a function in the martingale Hardy space $H_1(G_m)$, such that the partial sums of f are not bounded in $L_1(G_m)$ -norm, but partial sums S_n of the Vilenkin-Fourier series of a function $f \in L_1(G_m)$ convergence in measure [12].

Uniform convergence and some approximation properties of partial sums in $L_1(G_m)$ norms was investigate by Goginava [8] (see also [9]). Fine [3] has obtained sufficient conditions for the uniform convergence which are in complete analogy with the Dini-Lipschits conditions. Gulichev [13] has estimated the rate of uniform convergence of a Walsh-Fourier series using Lebesgue constants and modulus of continuity. Uniform convergence of subsequence of partial sums was investigate also in [7]. This problem has been considered for Vilenkin group G_m by Fridli [4], Blahota [2] and Gát [6].

It is also known that subsequence S_{n_k} is bounded from $L_1(G_m)$ to $L_1(G_m)$ if and only if n_k has uniformly bounded variation and subsequence of partial sums S_{M_n} is bounded from the martingale Hardy space $H_p(G_m)$ to the Lebesgue space $L_p(G_m)$, for all $p > 0$. In this paper we shall prove very unexpected fact:

There exists a martingale $f \in H_p(G_m)$ ($0 < p < 1$), such that

$$\sup_{n \in \mathbb{N}} \|S_{M_n+1}f\|_{L_{p,\infty}} = \infty.$$

The reason of divergence of $S_{M_n+1}f$ is that when $0 < p < 1$ the Fourier coefficients of $f \in H_p(G_m)$ are not bounded (See [17]).

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In Gát [5] the following strong convergence result was obtained for all $f \in H_1(G_m)$:

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{\|S_k f - f\|_1}{k} = 0,$$

where $S_k f$ denotes the k -th partial sum of the Vilenkin-Fourier series of f . (For the trigonometric analogue see Smith [16], for the Walsh system see Simon [14]). For the Vilenkin system Simon [15] proved that there is an absolute constant c_p , depending only on p , such that

$$(1) \quad \sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p,$$

for all $f \in H_p(G_m)$, where $0 < p < 1$. The author [18] proved that for any nondecreasing function $\Phi : \mathbb{N} \rightarrow [1, \infty)$, satisfying the condition $\lim_{n \rightarrow \infty} \Phi(n) = +\infty$, there exists a martingale $f \in H_p(G_m)$, such that

$$(2) \quad \sum_{k=1}^{\infty} \frac{\|S_k f\|_{L^{p,\infty}}^p \Phi(k)}{k^{2-p}} = \infty, \text{ for } 0 < p < 1.$$

Strong convergence theorems of two-dimensional partial sums was investigated by Weisz [23], Goginava [10], Gogoladze [11], Tephnadze [19].

The main aim of this paper is to investigate weighted maximal operators of partial sums of Vilenkin-Fourier series. We also use this results to prove some approximation and strong convergence theorems on the martingale Hardy spaces $H_p(G_m)$, when $0 < p \leq 1$.

2. DEFINITIONS AND NOTATIONS

Let \mathbb{N}_+ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$.

Let $m := (m_0, m_1, \dots)$ denote a sequence of the positive integers not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k .

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j} 's.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If the sequence $m := (m_0, m_1, \dots)$ is bounded than G_m is called a bounded Vilenkin group, else it is called an unbounded one.

The elements of G_m represented by sequences

$$x := (x_0, x_1, \dots, x_j, \dots) \quad (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, \quad n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \setminus I_n$.

It is evident

$$(3) \quad \overline{I_N} = \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1}.$$

If we define the so-called generalized number system based on m in the following way

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N})$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in Z_{m_k}$ ($j \in \mathbb{N}$) and only a finite number of n_j 's differ from zero. Let $|n| := \max \{j \in \mathbb{N}, n_j \neq 0\}$.

Denote by $L_1(G_m)$ the usual (one dimensional) Lebesgue space.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system.

At first define the complex valued function $r_k(x) : G_m \rightarrow \mathbb{C}$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k) \quad (t^2 = -1, \quad x \in G_m, \quad k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1, 20].

Now we introduce analogues of the usual definitions in Fourier-analysis.

If $f \in L_1(G_m)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system ψ in the usual manner:

$$\begin{aligned} \widehat{f}(k) &: = \int_{G_m} f \overline{\psi_k} d\mu, \quad (k \in \mathbb{N}), \\ S_n f &: = \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad (n \in \mathbb{N}_+, \quad S_0 f := 0), \\ D_n &: = \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+). \end{aligned}$$

Recall that (see [1])

$$(4) \quad D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases}$$

and

$$(5) \quad D_n(x) = \psi_n(x) \left(\sum_{j=0}^{\infty} D_{M_j}(x) \sum_{u=m_j-n_j}^{m_j-1} r_j^u(x) \right).$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f|^p d\mu \right)^{1/p} \quad (0 < p < \infty).$$

The space $L_{p,\infty}(G_m)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\lambda > 0} \lambda \mu(f > \lambda)^{1/p} < +\infty.$$

The σ -algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by F_n ($n \in \mathbb{N}$). Denote by $f = (f_n, n \in \mathbb{N})$ a martingale with respect to F_n ($n \in \mathbb{N}$). (for details see e.g. [21]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|.$$

In case $f \in L_1(G_m)$, the maximal functions are also given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) d\mu(u) \right|$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales, for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

The dyadic Hardy martingale spaces $H_p(G_m)$ for $0 < p \leq 1$ have an atomic characterization. Namely the following theorem is true (see [24]):

Theorem W: A martingale $f = (f_n, n \in \mathbb{N})$ is in $H_p(G_m)$ ($0 < p \leq 1$) if and only if there exists a sequence $(a_k, k \in \mathbb{N})$ of p -atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of a real numbers such that for every $n \in \mathbb{N}$

$$(6) \quad \sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f_n$$

and

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover, $\|f\|_{H_p} \sim \inf (\sum_{k=0}^{\infty} |\mu_k|^p)^{1/p}$, where the infimum is taken over all decomposition of f of the form (6).

Let $X = X(G_m)$ denote either the space $L_1(G_m)$, or the space of continuous functions $C(G_m)$. The corresponding norm is denoted by $\|\cdot\|_X$. The modulus of continuity, when $X = C(G_m)$ and the integrated modulus of continuity, where $X = L_1(G_m)$ are defined by

$$\omega(1/M_n, f)_X = \sup_{h \in I_n} \|f(\cdot + h) - f(\cdot)\|_X.$$

The concept of modulus of continuity in $H_p(G_m)$ ($0 < p \leq 1$) can be defined in following way

$$\omega(1/M_n, f)_{H_p(G_m)} := \|f - S_{M_n} f\|_{H_p(G_m)}.$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in \mathbb{N})$ is a martingale.

If $f = (f_n, n \in \mathbb{N})$ is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \rightarrow \infty} \int_{G_m} f_k(x) \overline{\Psi_i}(x) d\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as the martingale $(S_{M_n}(f) : n \in \mathbb{N})$ obtained from f .

For the martingale f we consider maximal operators

$$S^* f : = \sup_{n \in \mathbb{N}} |S_n f|,$$

$$\widetilde{S}_p^* f : = \sup_{n \in \mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1} \log^{[p]}(n+1)}, \quad 0 < p \leq 1,$$

where $[p]$ denotes integer part of p .

A bounded measurable function a is p -atom, if there exist a dyadic interval I , such that

$$\int_I a d\mu = 0, \quad \|a\|_{\infty} \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.$$

3. FORMULATION OF MAIN RESULTS

Theorem 1. a) Let $0 < p \leq 1$. Then the maximal operator \widetilde{S}_p^* is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$.

b) Let $0 < p \leq 1$ and $\varphi : \mathbb{N}_+ \rightarrow [1, \infty)$ be a nondecreasing function satisfying the condition

$$(7) \quad \lim_{n \rightarrow \infty} \frac{(n+1)^{1/p-1} \log^{[p]}(n+1)}{\varphi(n)} = +\infty.$$

Then

$$\sup_{n \in \mathbb{N}} \left\| \frac{S_n f}{\varphi(n)} \right\|_{L_{p,\infty}(G_m)} = \infty, \text{ for } 0 < p < 1$$

and

$$\sup_{n \in \mathbb{N}} \left\| \frac{S_n f}{\varphi(n)} \right\|_1 = \infty.$$

Corollary 1. (Simon [15]) Let $0 < p < 1$ and $f \in H_p(G_m)$. Then there is an absolute constant c_p , depends only p , such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p.$$

Theorem 2. Let $0 < p \leq 1$, $f \in H_p(G_m)$ and $M_k < n \leq M_{k+1}$. Then there is an absolute constant c_p , depends only p , such that

$$\|S_n(f) - f\|_{H_p(G_m)} \leq c_p n^{1/p-1} \lg^{[p]} n \omega\left(\frac{1}{M_k}, f\right)_{H_p(G_m)}.$$

Theorem 3. a) Let $0 < p < 1$, $f \in H_p(G_m)$ and

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} = o\left(\frac{1}{M_n^{1/p-1}}\right), \text{ as } n \rightarrow \infty.$$

Then

$$\|S_k(f) - f\|_{L_{p,\infty}(G_m)} \rightarrow 0, \text{ when } k \rightarrow \infty.$$

b) For every $p \in (0, 1)$ there exists martingale $f \in H_p(G_m)$, for which

$$\omega\left(\frac{1}{M_{2n}}, f\right)_{H_p(G_m)} = O\left(\frac{1}{M_{2n}^{1/p-1}}\right), \text{ as } n \rightarrow \infty$$

and

$$\|S_k(f) - f\|_{L_{p,\infty}(G_m)} \not\rightarrow 0, \text{ when } k \rightarrow \infty$$

Theorem 4. Let $f \in H_1(G_m)$ and

$$\omega\left(\frac{1}{M_n}, f\right)_{H_1(G_m)} = o\left(\frac{1}{n}\right), \text{ as } n \rightarrow \infty.$$

Then

$$\|S_k(f) - f\|_1 \rightarrow 0, \text{ when } k \rightarrow \infty.$$

b) There exists martingale $f \in H_1(G_m)$ for which

$$\omega\left(\frac{1}{M_{2M_n}}, f\right)_{H_1(G_m)} = O\left(\frac{1}{M_n}\right), \text{ as } n \rightarrow \infty$$

and

$$\|S_k(f) - f\|_1 \not\rightarrow 0 \text{ when } k \rightarrow \infty.$$

4. AUXILIARY PROPOSITIONS

Lemma 1. [22] *Suppose that an operator T is sublinear and for some $0 < p \leq 1$*

$$\int_I |Ta|^p d\mu \leq c_p < \infty,$$

for every p -atom a , where I denote the support of the atom. If T is bounded from L_∞ to L_∞ . Then

$$\|Tf\|_p \leq c_p \|f\|_{H_p(G_m)}.$$

Lemma 2. [17] *Let $n \in \mathbb{N}$ and $x \in I_s \setminus I_{s+1}$, $0 \leq s \leq N-1$. Then*

$$\int_{I_N} |D_n(x-t)| d\mu(t) \leq \frac{cM_s}{M_N}.$$

5. PROOF OF THE THEOREMS

Proof of Theorem 1. Since \tilde{S}_p^* is bounded from $L_\infty(G_m)$ to $L_\infty(G_m)$ by Lemma 1 we obtain that the proof of theorem 1 will be complete, if we show that

$$\int_{I_N} \left| \tilde{S}_p^* a(x) \right|^p d\mu(x) \leq c < \infty, \text{ when } 0 < p \leq 1,$$

for every p -atom a , where I denotes the support of the atom.

Let a be an arbitrary p -atom with support I and $\mu(I) = M_N$. We may assume that $I = I_N$. It is easy to see that $S_n(a) = 0$ when $n \leq M_N$. Therefore we can suppose that $n > M_N$.

Since $\|a\|_\infty \leq M_N^{1/p}$ we can write

$$\begin{aligned} (8) \quad |S_n(a)| &\leq \int_{I_N} |a(t)| |D_n(x-t)| d\mu(t) \\ &\leq \|a\|_\infty \int_{I_N} |D_n(x-t)| d\mu(t) \leq M_N^{1/p} \int_{I_N} |D_n(x-t)| d\mu(t). \end{aligned}$$

Let $0 < p < 1$ and $x \in I_s \setminus I_{s+1}$. From Lemma 2 we get

$$(9) \quad \frac{|S_n a(x)|}{\log^{[p]}(n+1)(n+1)^{1/p-1}} \leq \frac{cM_N^{1/p-1}M_s}{\log^{[p]}(n+1)(n+1)^{1/p-1}}.$$

Combining (3) and (9) we obtain

$$(10) \quad \int_{I_N} \left| \tilde{S}_p^* a(x) \right|^p d\mu(x) = \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} \left| \tilde{S}_p^* a(x) \right|^p d\mu(x) \\ \leq \frac{cM_N^{1-p}}{\log^{[p]p}(n+1)(n+1)^{1-p}} \sum_{s=0}^{N-1} \frac{M_s^p}{M_s} \leq \frac{cM_N^{1-p} N^{[p]}}{\log^{p[p]}(n+1)(n+1)^{1-p}} < c_p < \infty.$$

Let $0 < p < 1$. Applying (8), (10) and Theorem W we have

$$(11) \quad \sum_{k=M_N}^{\infty} \frac{\|S_k a\|_p^p}{k^{2-p}} \leq \sum_{k=M_N}^{\infty} \frac{1}{k} \int_{I_N} \left| \frac{S_k a(x)}{k^{1/p-1}} \right|^p d\mu(x) \\ + \sum_{k=M_N}^{\infty} \frac{M_N}{k^{2-p}} \int_{I_N} \left(\int_{I_N} |D_k(x-t)| d\mu(t) \right)^p d\mu(x) \\ \leq c_p M_N^{1-p} \sum_{k=M_N}^{\infty} \frac{1}{k^{2-p}} + c_p M_N^{1-p} \sum_{k=M_N}^{\infty} \frac{\log^p k}{k^{2-p}} \leq c_p < \infty.$$

Which complete the proof of corollary 1.

Let prove second part of Theorem 1. Let

$$f_{n_k}(x) = D_{M_{2n_k+1}}(x) - D_{M_{2n_k}}(x).$$

It is evident

$$\hat{f}_{n_k}(i) = \begin{cases} 1, & \text{if } i = M_{2n_k}, \dots, M_{2n_k+1} - 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then we can write

$$(12) \quad S_i f_{n_k}(x) = \begin{cases} D_i(x) - D_{M_{2n_k}}(x), & \text{if } i = M_{2n_k} + 1, \dots, M_{2n_k+1} - 1, \\ f_{n_k}(x), & \text{if } i \geq M_{2n_k+1}, \\ 0, & \text{otherwise.} \end{cases}$$

From (4) we get

$$(13) \quad \|f_{n_k}\|_{H_p(G_m)} = \left\| \sup_{n \in \mathbb{N}} S_{M_n}(f_{n_k}) \right\|_p = \|D_{M_{2n_k+1}} - D_{M_{2n_k}}\|_p \leq c_p M_{2n_k}^{1-1/p}.$$

Let $0 < p < 1$. Under condition (7) there exists positive integers n_k such that

$$\lim_{k \rightarrow \infty} \frac{(M_{2n_k} + 2)^{1/p-1}}{\varphi(M_{2n_k} + 2)} = \infty, \quad 0 < p < 1.$$

Applying (4), (5) and (12) we can write

$$\frac{|S_{M_{2n_k}+1} f_{n_k}|}{\varphi(M_{2n_k} + 2)} = \frac{|D_{M_{2n_k}+1} - D_{M_{2n_k}}|}{\varphi(M_{2n_k} + 2)} = \frac{|w_{M_{2n_k}}|}{\varphi(M_{2n_k} + 2)} = \frac{1}{\varphi(M_{2n_k} + 2)}.$$

Hence we can write:

$$(14) \quad \mu \left\{ x \in G_m : \frac{|S_{M_{2n_k}+1} f_{n_k}(x)|}{\varphi(M_{2n_k}+2)} \geq \frac{1}{\varphi(M_{2n_k}+2)} \right\} = 1.$$

Combining (13) and (14) we have

$$\begin{aligned} & \frac{\frac{1}{\varphi(M_{2n_k}+2)} \left(\mu \left\{ x \in G_m : \frac{|S_{M_{2n_k}+1} f_{n_k}(x)|}{\varphi(M_{2n_k}+2)} \geq \frac{1}{\varphi(M_{2n_k}+2)} \right\} \right)^{1/p}}{\|f_{n_k}(x)\|_{H_p}} \\ & \geq \frac{1}{\varphi(M_{2n_k}+2) M_{2n_k}^{1-1/p}} = \frac{(M_{2n_k}+2)^{1/p-1}}{\varphi(M_{2n_k}+2)} \rightarrow \infty, \text{ when } k \rightarrow \infty. \end{aligned}$$

Now consider the case when $p = 1$. Under condition (7) there exists $\{n_k : k \geq 1\}$, such that

$$\lim_{k \rightarrow \infty} \frac{\log q_{n_k}}{\varphi(q_{n_k})} = \infty.$$

Let $q_{n_k} = M_{2n_k} + M_{2n_k-2} + M_2 + M_0$ and $x \in I_{2s} \setminus I_{2s+1}$, $s = 0, \dots, n_k$.

Combining (4) and (5) we have

$$\begin{aligned} |D_{q_{n_k}}(x)| & \geq |D_{M_{2s}}(x)| - \left| \sum_{l=0}^{s-2} r_{2l}^{m_{2l}-1}(x) D_{M_{2l}}(x) \right| \\ & \geq M_{2s} - \sum_{l=0}^{s-2} M_{2l} \geq M_{2s} - M_{2s-1} \geq \frac{M_{2s}}{2}. \end{aligned}$$

Hence

$$(15) \quad \int_{G_m} |D_{q_{n_k}}(x)| d\mu(x) \geq \frac{1}{2} \sum_{s=0}^{n_k} \int_{I_{2s} \setminus I_{2s+1}} M_{2s} d\mu(x) \geq c \sum_{s=0}^{n_k} 1 \geq cn_k.$$

From (12), (13) and (15) we have

$$\begin{aligned} & \frac{1}{\|f_{n_k}(x)\|_{H_1(G_m)}} \int_{G_m} \frac{|S_{q_{n_k}} f_{n_k}(x)|}{\varphi(q_{n_k})} d\mu(x) \\ & \geq \frac{1}{\|f_{n_k}(x)\|_{H_1(G_m)}} \left(\int_{G_m} \frac{|D_{q_{n_k}}(x)|}{\varphi(q_{n_k})} d\mu(x) - \int_{G_m} \frac{|D_{M_{2n_k}}(x)|}{\varphi(q_{n_k})} d\mu(x) \right) \\ & \geq \frac{c}{\varphi(q_{n_k})} (\log q_{n_k} - 1) \geq \frac{c \log q_{n_k}}{\varphi(q_{n_k})} \rightarrow \infty, \quad \text{when } k \rightarrow \infty. \end{aligned}$$

Which complete the proof of theorem 1.

Proof of Theorem 2. Let $0 < p \leq 1$ and $M_k < n \leq M_{k+1}$. Using Theorem 1 we have

$$\|S_n f\|_p \leq c_p n^{1/p-1} \log^{[p]} n \|f\|_{H_p(G_m)}.$$

Hence

$$\begin{aligned} \|S_n f - f\|_p^p &\leq \|S_n f - S_{M_k} f\|_p^p + \|S_{M_k} f - f\|_p^p = \|S_n (S_{M_k} f - f)\|_p^p \\ &+ \|S_{M_k} f - f\|_p^p \leq c_p (n^{1-p} + 1) \log^{p[p]} n \omega^p \left(\frac{1}{M_k}, f \right)_{H_p(G_m)} \end{aligned}$$

and

$$(16) \quad \|S_n f - f\|_p \leq c_p n^{1/p-1} \log^{[p]} n \omega \left(\frac{1}{M_k}, f \right)_{H_p(G_m)}.$$

Proof of Theorem 3. Let $0 < p < 1$, $f \in H_p(G_m)$ and

$$\omega \left(\frac{1}{M_{2n}}, f \right)_{H_p(G_m)} = o \left(\frac{1}{M_{2n}^{1/p-1}} \right), \text{ as } n \rightarrow \infty.$$

Using (16) we immediately get

$$\|S_n f - f\|_p \rightarrow \infty, \text{ when } n \rightarrow \infty.$$

Let proof of second part of theorem 3. We set

$$a_k(x) = \frac{M_{2k}^{1/p-1}}{\lambda} (D_{M_{2k+1}}(x) - D_{M_{2k}}(x)),$$

where $\lambda = \sup_{n \in \mathbb{N}} m_n$ and

$$f_A(x) = \sum_{i=0}^A \frac{\lambda}{M_{2i}^{1/p-1}} a_i(x).$$

Since

$$(17) \quad S_{M_A} a_k(x) = \begin{cases} a_k(x), & 2k \leq A, \\ 0, & 2k > A, \end{cases}$$

and

$$\text{supp}(a_k) = I_{2k}, \quad \int_{I_{2k}} a_k d\mu = 0, \quad \|a_k\|_\infty \leq M_{2k}^{1/p-1} \cdot M_{2k} = M_{2k}^{1/p} = (\text{supp } a_k)^{-1/p},$$

if we apply Theorem W we conclude that $f \in H_p$.

It is easy to show that

$$\begin{aligned}
 (18) \quad & f - S_{M_n} f \\
 &= \left(f^{(1)} - S_{M_n} f^{(1)}, \dots, f^{(n)} - S_{M_n} f^{(n)}, \dots, f^{(n+k)} - S_{M_n} f^{(n+k)} \right) \\
 &= \left(0, \dots, 0, f^{(n+1)} - f^{(n)}, \dots, f^{(n+k)} - f^{(n)}, \dots \right) \\
 &= \left(0, \dots, 0, \sum_{i=n}^k \frac{a_i(x)}{M_i^{1/p-1}}, \dots \right), \quad k \in \mathbb{N}_+
 \end{aligned}$$

is martingale. Using (18) we get

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p} \leq \sum_{i=[n/2]+1}^{\infty} \frac{1}{M_{2i}^{1/p-1}} = O\left(\frac{1}{M_n^{1/p-1}}\right).$$

where $[n/2]$ denotes integer part of $n/2$. It is easy to show that

$$(19) \quad \widehat{f}(j) = \begin{cases} 1, & \text{if } j \in \{M_{2i}, \dots, M_{2i+1} - 1\}, \quad i = 0, 1, \dots \\ 0, & \text{if } j \notin \bigcup_{i=0}^{\infty} \{M_{2i}, \dots, M_{2i+1} - 1\}. \end{cases}$$

Using (19) we have

$$\begin{aligned}
 & \limsup_{k \rightarrow \infty} \|f - S_{M_{2k+1}-1}(f)\|_{L_{p,\infty}(G_m)} \\
 & \geq \limsup_{k \rightarrow \infty} \left(\|w_{M_{2k+1}-1}\|_{L_{p,\infty}(G_m)} - \left\| \sum_{i=k+1}^{\infty} (D_{M_{2i+1}} - D_{M_{2i}}) \right\|_{L_{p,\infty}(G_m)} \right) \\
 & \geq \limsup_{k \rightarrow \infty} \left(1 - c/M_{2k}^{1/p-1} \right) c > 0.
 \end{aligned}$$

Which complete the proof of Theorem 3.

Proof of Theorem 4. Analogously we can prove first part of Theorem 4. Let proof it's second part. We set

$$a_i(x) = D_{M_{2M_i+1}}(x) - D_{M_{2M_i}}(x)$$

and

$$f_A(x) = \sum_{i=1}^A \frac{a_i(x)}{M_i}.$$

Since

$$(20) \quad S_{M_A} a_k(x) = \begin{cases} a_k(x), & 2M_k \leq A, \\ 0, & 2M_k > A, \end{cases}$$

and

$$\text{supp}(a_k) = I_{2M_k}, \quad \int_{I_{2M_k}} a_k d\mu = 0, \quad \|a_k\|_{\infty} \leq M_{2M_k} = \mu(\text{supp } a_k),$$

if we apply Theorem W we conclude that $f \in H_1$.

It is easy to show that

$$(21) \quad \omega\left(\frac{1}{M_n}, f\right)_{H_1(G_m)} \leq \sum_{i=\lfloor \lg n/2 \rfloor}^{\infty} \frac{1}{M_i} = O\left(\frac{1}{n}\right),$$

where $\lfloor \lg n/2 \rfloor$ denotes integer part of $\lg n/2$. By simple calculation we get

$$(22) \quad \widehat{f}(j) = \begin{cases} \frac{1}{M_{2i}}, & \text{if } j \in \{M_{2M_i}, \dots, M_{2M_{i+1}} - 1\}, \quad i = 0, 1, \dots \\ 0, & \text{if } j \notin \bigcup_{i=0}^{\infty} \{M_{2M_i}, \dots, M_{2M_{i+1}} - 1\}. \end{cases}$$

Combining (15) and (22) we have

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \|f - S_{q_{M_k}}(f)\|_1 \\ & \geq \limsup_{k \rightarrow \infty} \left(\frac{1}{M_{2k}} \|D_{q_{M_k}}\|_1 - \frac{1}{M_{2k}} \|D_{M_{2M_{k+1}}}\|_1 - \left\| \sum_{i=k+1}^{\infty} \frac{D_{M_{2M_{i+1}}} - D_{M_{2M_i}}}{M_{2i}} \right\|_1 \right) \\ & \geq \limsup_{k \rightarrow \infty} \left(c - \sum_{i=k+1}^{\infty} \frac{1}{M_{2i}} - \frac{1}{M_{2k}} \right) \geq c > 0. \end{aligned}$$

Theorem 4 is proved.

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